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On the enumeration of the simple 3-polyhedra

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It will be proved that the simple 3-polyhedra with $f+1$ facets are obtained from all simple 3-polyhedra with f facets by 2-face splits. The numbers of combinatorial types of simple 3-polyhedra with up to 15 facets are stated with respect to their automorphism group order.

1. Basic definitions

The characterization of the facial structure of polyhedra has a long tradition in combinatorial geometry. Steinitz & Rademacher's (1934) fundamental theorem asserts that

Theorem of Steinitz. Every finite 3-connected planar graph is isomorphic to the edge graph of a 3-polyhedron.

Under a *3-polyhedron* P, we understand any bounded non-degenerate region of Euclidean space E^3 which is obtained by intersecting a finite number f of closed half spaces H_i,

$$\mathsf{P} = \bigcap_{i=1}^{f} \mathsf{H}_i. \qquad (1)$$

The half spaces H_i are convex, hence P is convex too. We denote by *vertices*, *edges* and *facets* the 0-, 1- and 2-faces of P, respectively. The 3-face is P itself.

Among the k-faces of P, $0 \le k \le 3$, there exists a partial ordering with respect to the inclusion operation. The k-faces, together with the empty set, determine by inclusion the *face lattice* $\mathcal{L}(\mathsf{P})$ that is, for any two faces F and F' of $\mathcal{L}(\mathsf{P})$, the least upper bound is given by the k-face $\mathsf{F}_\vee \supset \mathsf{F} \cup \mathsf{F}'$ having least k. The greatest lower bound is given by the l-face $\mathsf{F}_\wedge = \mathsf{F} \cap \mathsf{F}'$.

The 3-polyhedron having least k-faces is the *3-simplex*. It consists of $\binom{4}{k+1}$ k-faces, $0 \le k \le 3$.

Two polyhedra P and P' are called *combinatorially equivalent*, $\mathsf{P}' \overset{\text{comb}}{\simeq} \mathsf{P}$, and belong to the same *combinatorial type*, if there exists a combinatorial isomorphism $\kappa : \mathcal{L}(\mathsf{P}) \to \mathcal{L}(\mathsf{P}')$. Any realization of a polyhedron that is combinatorial equivalent to P is called a *representative* of its combinatorial type.

If the combinatorial structure of a 3-polyhedron P is considered only, a half-space cut applied to P which results in P' can be represented by a transformation τ of the corresponding face lattice $\mathcal{L}(\mathsf{P}') = \tau \circ \mathcal{L}(\mathsf{P})$. We denote such a transformation as a *combinatorial cut*.

In order to classify the combinatorial types, we use the *unified polyhedron scheme* which was introduced by Engel (1981), and then generalized (Engel, 1991*a*,*b*). A *flag* is defined to be a subseries of successively subordinated k-faces of $\mathcal{L}(\mathsf{P})$,

$$\text{0-face} \subset \text{1-face} \subset \text{2-face} \subset \mathsf{P}.$$

Given any such flag, we can consecutively number all k-faces of $\mathcal{L}(\mathsf{P})$ in a unique way. Next, we set up the polyhedron scheme by writing down for each facet in their consecutive order the numbers of all its subordinated vertices in increasing order. This polyhedron scheme depends only on the chosen initial flag. We write the polyhedron schemes for each possible flag, and put them into classes of identical schemes. The classes are lexicographically ordered, and a representative of the first class is taken as the unified polyhedron scheme of P. Each class contains the same number of identical schemes which equals the order of the *combinatorial automorphism group* of $\mathcal{L}(\mathsf{P})$. The unified polyhedron scheme gives a unique characterization of $\mathcal{L}(\mathsf{P})$ and, hence, of its combinatorial type. On the other hand, we can generate $\mathcal{L}(\mathsf{P})$ from its polyhedron scheme.

It was proved by Mani (1971) that

Theorem of Mani. For each 3-polyhedron P there exists a realization P' such that
(i) $\mathsf{P}' \overset{\text{comb}}{\simeq} \mathsf{P}$;
(ii) the point group of P' is isomorphous to the combinatorial automorphism group of $\mathcal{L}(\mathsf{P})$.

A 3-polyhedron P is called *simple*, if in every vertex of it exactly three facets meet, and thus in each vertex three edges meet. By *valence* of the vertex, we denote the number of edges meeting in a vertex. In what follows, we will mainly consider simple 3-polyhedra. Given the number f of facets of a simple 3-polyhedron, the numbers v and e of vertices and edges, respectively, are readily obtained from Euler's relation.

$$v = 2f - 4, \quad e = 3f - 6. \qquad (2)$$

Let f_i be the number of 2-faces of P that have subordinated e_i edges, $i = 1, \ldots, r$. The 2-*subordination symbol* is defined by

$$e_{1_{f_1}} e_{2_{f_2}} \ldots e_{r_{f_r}}, \qquad (3)$$

with $e_1 < e_2 < \ldots < e_r$. It holds that

$$f = \sum_{i=1}^{r} f_i.$$

In Engel (1982), the combinatorial types of 3-polyhedra with up to 11 facets were calculated, starting from a 3-simplex, by cuts and edge contractions. The combinatorial types of simple 3-polyhedra with $f+1$ facets were derived from all combinatorial types of 3-polyhedra with f facets which have at most one vertex of valence >3. In this note, we present an algorithm, called the *split algorithm*, which allows the determination of the combinatorial types of simple 3-polyhedra with $f+1$ facets directly from the simple ones with f facets using 2-face splits only.

In order to determine a 3-polyhedron having $f+1$ facets from a 3-polyhedron P having f facets, we use half-space intersections which cut off some part of a facet of P. We only consider general cuts that contain no vertices of P. The simplicity of P requires that edges of P are cut in inner points only. As a *k-face cut*, we denote the cutting off of a complete k-face, $k=0,1$. As a *2-face split*, we denote the cutting off of n consecutive vertices of a single facet having s vertices, $0<n<s$. A 2-face split is called *proper* if $n>2$. For $n=1,2$, the 2-face splits correspond to 0- and 1-face cuts, respectively. In order to economize on the calculations, it is useful to treat the 0- and 1-face cuts separately. A cut dissects P into two simple 3-polyhedra,

$$\mathsf{P}=\mathsf{P}'\sqcup\Delta, \qquad (4)$$

where $\sqcup$ is the standard notation for the pairwise interior disjoint union of P' and Δ. Let P' be the resulting polyhedron, and let Δ be the piece cut off. When cutting off n consecutive vertices of a single facet, $n>2$, we denote Δ to be an *n-blade*.

Let $\mathsf{F}\subset\mathsf{P}$ be a 2-face with s vertices $v_1,\ldots,v_s$. We specify n consecutive vertices from F, $v_{l_1}-v_{l_2}-\ldots-v_{l_n}$, $0<n<s$. As *complement cut* in F, we denote the cutting off of the other $s-n$ consecutive vertices of F.

In any dimension $d\geq 3$, the k-face cuts, $0\leq k\leq d-2$, and the 2-face splits were denoted in Engel (1991*a*) as *free cuts* because they can always be performed, independently of the particular shape of any d-polytope.

Besides the 2-face splits, we will make use of the *crossing operation* introduced by Eberhard (1891) which is justified by the theorem of Steinitz. For a simple 3-polyhedron P, consider the edge E_{12} having subordinated the vertices v_1 and v_2. Since P is simple, each vertex has valence 3. We denote the 3 edges radiating from v_1 to v_2, v_3 and v_4 by E_{12}, E_{13} and E_{14}, respectively, and those radiating from v_2 to v_5 and v_6 by E_{25} and E_{26}, respectively. The four sets of consecutive vertices $v_3-v_1-v_2-v_5$, $v_4-v_1-v_2-v_6$, $v_3-v_1-v_4$ and $v_5-v_2-v_6$ belong to the facets F_1, F_2, F_3 and F_4, respectively. We require the facets F_1 and F_2 to have subordinated more than three vertices each. As a crossing operation, we understand the following process: The edge E_{12} is contracted such that the subordinated vertices v_1 and v_2 coincide, and then both vertices are taken apart again but in a direction perpendicular to the original edge E_{12}. Thus, the new edge E'_{12} forms a cross with the original edge E_{12}. The transformed vertex v'_1 is connected by three edges to v'_2, v_3 and v_5, and the transformed vertex v'_2 is connected by three edges to v'_1, v_4 and v_6, respectively. By the crossing operation, the facets F_3 and F_4 become adjacent and their numbers of vertices increase by one, whereas the other two facets, F_1 and F_2, become disjoint, and their numbers of vertices decrease by one.

2. The split algorithm

The split algorithm is based on the following three theorems.

Theorem 1. Every 3-polyhedron can be obtained starting from a 3-simplex by successive half-space intersections.

Proof. Theorem 1 is a direct consequence of (1). Let P be a 3-polyhedron having f facets. Since P is bounded, there exists a 3-simplex T such that $\mathsf{P}\subseteq\mathsf{T}$. From the convexity of P, it follows that each facet, irrespective of the order of succession, determines a unique half-space intersection. Starting from a 3-simplex, there are needed f half-space intersections in order to determine P. □

The second theorem reduces all feasible cuts to 2-face splits only.

Theorem 2. Each simple 3-polyhedron with $f+1$ facets can be obtained from a simple 3-polyhedron with f facets by a 2-face split.

Proof. Let P be a simple 3-polyhedron with f facets.

(i) Trivially, each 2-face split gives a 3-polyhedron with $f+1$ facets.

(ii) Assume any cut of arbitrary complexity, subject to the requirement that no complete facet is cut off because otherwise the number of facets would not increase. Necessarily, each facet which is involved in the cut contains exactly one cut edge. The cut edges define a cut polygon C, which is homeomorphic to a circle. The cut dissects P into $\mathsf{P}=\mathsf{P}'\sqcup\Delta$. By the conditions of the cut, each facet of Δ has at least one common edge with C. By the theorem of Steinitz, we can transform Δ by successive crossings to an n-blade $\widehat{\Delta}$ without changing C. Now, $\widehat{\mathsf{P}}=\mathsf{P}'\sqcup\widehat{\Delta}$ is a simple 3-polytope with f facets and the half-space intersection is, as proposed, a 2-face split. □

The third theorem reduces the number of required 2-face splits.

Theorem 3. Let F be a facet of a simple 3-polyhedron P, and assume that F has s vertices. Let P' be the result of cutting off n consecutive vertices of F, $0<n<s$, and let P'' be the result of the corresponding complement cut in F. Then it holds that $\mathsf{P}''\overset{\text{comb}}{\simeq}\mathsf{P}'$.

Proof. By cutting off n consecutive vertices of F, $0<n<s$, the number of vertices of P' is increased by 2. The two additional vertices result from the cut edge which splits F. The complement cut in F has the same cut edge in F, and, therefore, the same two additional vertices result for P''. It follows that $\mathcal{L}(\mathsf{P}')$ and $\mathcal{L}(\mathsf{P}'')$ have the same connectivity and, hence, they are combinatorially equivalent. □

Table 1
The numbers of combinatorial types of simple 3-polyhedra with given order of automorphism group.

Order	4	5	6	7	8	9	10	11	12	13	14	15
1	–	–	–	–	2	16	137	970	6756	47030	331796	2382352
2	–	–	–	1	5	25	69	241	747	2377	7587	23994
3	–	–	–	–	–	–	1	1	1	12	12	20
4	–	–	1	1	3	5	13	27	68	118	266	439
6	–	–	–	2	–	1	6	5	7	28	27	25
8	–	–	–	–	1	–	4	–	10	–	19	–
10	–	–	–	–	–	–	–	–	1	–	–	–
12	–	1	–	–	1	2	–	4	3	–	9	10
16	–	–	–	–	–	–	1	–	–	–	2	–
20	–	–	–	1	–	–	–	–	–	–	–	–
24	1	–	–	–	2	–	1	–	–	–	1	–
28	–	–	–	–	–	1	–	–	–	–	–	–
32	–	–	–	–	–	–	1	–	–	–	–	–
36	–	–	–	–	–	–	–	1	–	–	–	–
40	–	–	–	–	–	–	–	–	1	–	–	–
44	–	–	–	–	–	–	–	–	–	1	–	–
48	–	–	1	–	–	–	–	–	–	–	3	–
52	–	–	–	–	–	–	–	–	–	–	–	1
120	–	–	–	–	–	–	–	–	1	–	–	–
Total	1	1	2	5	14	50	233	1249	7595	49566	339722	2406841
2-sub	1	1	2	5	13	33	85	199	445	947	1909	3713
Max.	1	1	1	1	2	3	11	47	186	762	4692	21203

By theorem 1, the split algorithm to determine the simple 3-polyhedra starts from a 3-simplex. We only need to execute a 0-face cut, because the 1-face cut is the corresponding complement cut, and by theorem 3 the resulting 3-polyhedra are combinatorially equivalent. Thus we get the unique combinatorial type of simple 3-polyhedron having 5 facets.

We now assume that all combinatorial types of simple 3-polyhedra with f facets are known, and we proceed to determine those with $f+1$ facets. For each combinatorial type of simple 3-polyhedra P with f-facets we perform the following three steps:

(i) For each vertex of P, we execute a 0-face cut.

(ii) For each edge of P, where both adjacent facets have at least four vertices, we perform a 1-face cut on that edge. Edges that belong to a triangle need not be considered because the cut would correspond to a complement cut of some 0-face cut in (i).

(iii) For each facet $\mathsf{F} \subset \mathsf{P}$ that has $s \geq 6$ vertices, we cut off all possible n-blades, $3 \leq n \leq \lfloor \frac{s}{2} \rfloor$, beginning at each of the s vertices of F ($\lfloor \alpha \rfloor$ is the standard notation for the largest integer less than or equal to α). By theorem 3, we have to consider at most $n = \lfloor \frac{s}{2} \rfloor$ consecutive vertices of F. For $3 < s < 6$, any 2- face split would correspond to a complement 0- or 1-face cut in (ii).

By theorem 2, this completes the list of necessary 2-face splits that are needed in order to obtain the combinatorial types of simple 3-polyhedra with $f+1$ facets from those with f facets.

3. Results

The split algorithm was implemented in C-language. In order to perform the 2-face splits, the half-space intersection algorithm of Engel (1986) was used which transforms $\mathcal{L}(\mathsf{P})$ into $\mathcal{L}(\mathsf{P}')$. In order to classify the combinatorial types of simple 3-polyhedra, the unified polyhedron scheme was calculated for each simple 3-polyhedron produced.

The simple 3-polyhedra with few facets are well known. The combinatorial types of 3-polyhedra with up to 6 facets were determined by Steiner (1829), and those with up to 8 facets by Kirkman (1862), and again by Hermes (1899) and Britton & Dunitz (1973). Fedorov (1893) determined the combinatorial types of simple 3-polyhedra with nine facets. However, owing to an insufficient classification scheme, his list included two doubles as was found by Engel (1994). Duijvestijn & Federico (1981) determined the combinatorial types of 3-polyhedra with up to ten facets. Bowen & Fisk (1967) determined the combinatorial types of simple 3-polyhedra with 12 facets. In Engel (1982), the combinatorial types of 3-polyhedra with up to 11 facets were determined and, from these, the simple ones with 12 facets were obtained. All previous results could be confirmed. In Engel (1994), the 6386475 combinatorial types of 3-polyhedra with 12 facets were determined and, from these, the 49566 combinatorial types of simple 3-polyhedra with 13 facets were obtained. For simple 3-polyhedra with up to 13 facets, in a series of papers, Voytekhovsky (2001*a*,*b*) and Voytekhovsky & Stepenshchikov (2002) determined the symmetries of 3-polyhedra whose point groups are isomorphous to the combinatorial automorphism groups of the corresponding face lattices. With the split algorithm, we could confirm the known results on simple 3-polyhedra, and we further determined the combinatorial types of simple 3-polyhedra with 14 and 15 facets. The results are summarized in Table 1. In the table are given under the column order the combinatorial automorphism group orders of the edge graphs. The row 2-sub states the number of different 2-subordination symbols, and in the row max. are given the maximal numbers of combinatorial types of polyhedra having the same 2-subordination symbol. The unified polyhedron schemes for the combinatorial types of simple 3-polyhedra with up to 15 facets are available upon request.

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